

# ON THE EXISTENCE OF PERIODIC ORBITS FOR MAGNETIC SYSTEMS ON THE TWO-SPHERE

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**ABSTRACT.** We prove that there exist periodic orbits on almost all compact regular energy levels of a Hamiltonian function defined on a twisted cotangent bundle over the two-sphere. As a corollary, given any Riemannian two-sphere and a magnetic field on it, there exists a closed magnetic geodesic for almost all kinetic energy levels.

## 1. INTRODUCTION

We consider the cotangent bundle of the 2-sphere together with the bundle projection  $\pi: T^*S^2 \rightarrow S^2$ . The Liouville 1-form  $\lambda$  on  $T^*S^2$  is defined by  $\lambda_u = u \circ T\pi$  for all co-vectors  $u \in T^*S^2$ . The differential  $d\lambda$  provides  $T^*S^2$  with the canonical symplectic structure. For any 2-form  $\sigma$  on  $S^2$ , which is closed for dimensional reasons, the so-called **twisted symplectic form**

$$\omega_\sigma = d\lambda + \pi^*\sigma$$

is defined. The equations of motion of the Hamiltonian system given by  $\omega_\sigma$  and the sum of kinetic and potential energy describes the motion of a charged particle on  $S^2$  under the influence of the force field induced by the potential and the magnetic field induced by  $\sigma$ . We call  $\sigma$  the **magnetic form** and  $(T^*S^2, \omega_\sigma)$  a **twisted cotangent bundle**.

Let  $H$  be a Hamiltonian function on a twisted cotangent bundle over  $S^2$ . For any compact regular energy surface  $\{H = E\}$  we find  $\varepsilon > 0$  such that all energy levels of energy between  $E - \varepsilon$  and  $E + \varepsilon$  are compact and regular. We say that  $\{H = E\}$  has the **almost existence property** if for all such  $\varepsilon > 0$  the set of energy values in  $(E - \varepsilon, E + \varepsilon)$ , for which the corresponding energy surface of  $H$  admits a periodic solution, has measure  $2\varepsilon$ .

**Theorem 1.1.** *Any compact regular energy surface of a Hamiltonian function on a twisted cotangent bundle over  $S^2$  has the almost existence property.*

By passage to the double cover the theorem implies the almost existence property for Hamiltonian functions on twisted cotangent bundles over the real projective plane.

We remark that the third homology group of  $T^*S^2$  vanishes. Therefore, each compact connected orientable hypersurface  $M$  in  $T^*S^2$  is the boundary of a relatively compact domain  $D$ . By the results of Hofer-Zehnder [14, 15] and Struwe [32] the almost existence property for  $M$ , viewed as the regular level set of a function,

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follows provided that the Hofer-Zehnder capacity of  $D$  is finite. For an alternative argument we refer to or Macarini-Schlenk [23], which only requires finiteness of the Hofer-Zehnder capacity of a neighbourhood of  $M$ . Therefore, Theorem 1.1 is implied by:

**Theorem 1.2.** *Each open and relatively compact subset of  $(T^*S^2, \omega_\sigma)$  has finite Hofer-Zehnder capacity.*

We remark that the Hofer-Zehnder capacity of any twisted cotangent bundle (of dimension  $2n$ ) is infinite as the cotangent bundle of  $\mathbb{R}^n$  embeds symplectically as observed by Lu [20, Theorem D(ii)].

Hamiltonian systems on twisted cotangent bundles of closed manifolds were first introduced by Arnol'd in [3], where he considers as Hamiltonian function the kinetic energy with respect to some Riemannian metric. These systems are significant since they model many physical phenomena in classical mechanics: a charged particle under the influence of a magnetic force, a rigid body in an axially symmetric field. At the beginning of the Eighties, Novikov got interested in such systems from the point of view of periodic trajectories, see [29, 28]. This direction of research was later continued by Taĭmanov in a series of works, see [33] for a survey.

Starting with the work of Ginzburg [11] and Polterovich [30] symplectic approaches to the existence of periodic orbits proved also to be effective, see also Macarini [22]. Among the tools used for this purpose the Hofer-Zehnder capacity, with which we are concerned here, plays a predominant role. In the setting of twisted cotangent bundles it was studied by Lu [20, Theorem E], Ginzburg-Gürel [12, Section 2.3], Schlenk [31, Section 3.3], Frauenfelder-Schlenk [10, Theorem 4.B], Cieliebak-Frauenfelder-Paternain [6], and Irie [16, 17]. However, almost existence for high energy levels and non-exact magnetic forms was left open for the two-sphere.

Finally, we would like to mention the groundbreaking work of Contreras [7] on autonomous Lagrangian systems, which paved the way to tremendous advancements for the problem of closed orbits on twisted cotangent bundles, see Merry [27] and [4]. For multiplicity results when the base manifold is a surface see also Abbondandolo-Macarini-Paternain [1] as well as [2, 5].

The idea of the proof of Theorem 1.2 is to show that the twisted symplectic form can be interpolated to the canonical Liouville symplectic form. A compact neighbourhood of the zero-section that supports the interpolation in turn embeds symplectically into  $S^2 \times S^2$  provided with a split symplectic form. The Hofer-Zehnder capacity of such a symplectic manifold is finite, so that the result will follow from the monotonicity property of the capacity.

## 2. THE CUT OFF LEMMA

That  $\omega_\sigma$  is indeed a symplectic form follows from the local description

$$\omega_\sigma = dp_i \wedge dq^i + \sigma_{jk}(q) dq^j \wedge dq^k$$

with respect to canonical  $(q, p)$ -coordinates. In fact, the volume forms  $\omega_\sigma \wedge \omega_\sigma$  and  $d\lambda \wedge d\lambda$  are the same. Moreover, the canonical Liouville vector field  $Y$ , which equals  $p_i \partial_{p_i}$  locally, preserves the twisting  $\pi^* \sigma$ . In fact,  $i_Y \pi^* \sigma = 0$ .

**Lemma 2.1.** *For any compact subset  $K$  of  $T^*S^2$  there exists a symplectic form on  $T^*S^2$  that coincides with  $\omega_\sigma$  in a neighbourhood of  $K$  and with  $d\lambda$  in a neighbourhood of the end.*

*Proof.* We provide  $S^2$  with the metric induced by the inclusion into  $\mathbb{R}^3$ . This allows us to identify the unit co-sphere bundle of  $S^2$  with  $\mathbb{R}P^3$ . The restriction of  $\lambda$  to the tangent bundle of  $\mathbb{R}P^3$  induces a contact form  $\alpha$  on  $\mathbb{R}P^3$ . The set of all non-zero co-vectors  $u \in T^*S^2$  can be identified with the symplectisation

$$(\mathbb{R} \times \mathbb{R}P^3, d(e^t \alpha))$$

symplectically via the map

$$u \mapsto (\ln |u|, u/|u|),$$

which sends the flow lines of  $Y$  to the flow lines of  $\partial_t$ ,  $t \in \mathbb{R}$ , and hence  $\lambda$  to  $e^t \alpha$ .

We claim that  $\pi^* \sigma$  restricted to the symplectisation of  $\mathbb{R}P^3$  has a primitive  $\tau$  that is a pull back along the inclusion  $\mathbb{R}P^3 \equiv \{0\} \times \mathbb{R}P^3 \subset \mathbb{R} \times \mathbb{R}P^3$ . Indeed, write

$$\pi^* \sigma = a dt \wedge \gamma_t + \eta_t$$

for the image of  $\pi^* \sigma$  on  $\mathbb{R} \times \mathbb{R}P^3$ , where  $a$  is a function on  $\mathbb{R} \times \mathbb{R}P^3$  and  $\gamma_t$ , resp.,  $\eta_t$  is a  $t$ -parameter family of 1-, resp., 2-forms on  $\mathbb{R}P^3$ . As observed at the beginning of the section inner multiplication of the twisting  $\pi^* \sigma$  by  $Y \equiv \partial_t$  yields zero, so that  $a \gamma_t = 0$  and, hence,  $\pi^* \sigma = \eta_t$  on the symplectisation. Therefore, we get  $0 = \dot{\eta}_t \wedge dt + d\eta_t$  because  $\sigma$  is closed, where the dot indicates the time derivative. This implies  $\dot{\eta}_t = 0$  again by using  $\partial_t$ , and hence  $d\eta_t = 0$ . In other words,  $\pi^* \sigma = \eta$  for a closed 2-form  $\eta$  on  $\mathbb{R}P^3$ , which must have a primitive  $\tau$  because the second de Rham cohomology of  $\mathbb{R}P^3$  vanishes. In conclusion,  $\pi^* \sigma = d\tau$  on  $\mathbb{R} \times \mathbb{R}P^3$ .

Let  $t_0$  be a sufficiently large real number. Let  $f$  be a cut off function on  $\mathbb{R}$ , that is identically 1 on  $(-\infty, t_0]$ , resp., 0 on  $[t_0 + R, \infty)$  for some positive real number  $R$ , and has derivative  $\dot{f}$  with values in the interval  $(-2/R, 0]$ . In order to prove the lemma we will show that

$$d(e^t \alpha) + d(f\tau)$$

is a symplectic form for sufficiently large  $R$ . Because the volume forms induced by  $d(e^t \alpha)$  and the twisted  $d(e^t \alpha) + d\tau$  coincide their difference  $-2e^t dt \wedge \alpha \wedge d\tau$  must vanish. Hence

$$\alpha \wedge d\tau = 0.$$

Therefore, the square of  $d(e^t \alpha) + d(f\tau)$  equals

$$2dt \wedge (e^{2t} \alpha \wedge d\alpha + \dot{f} e^t \tau \wedge d\alpha + \dot{f} f \tau \wedge d\tau).$$

This is indeed a volume form provided that the 3-forms

$$e^t \alpha \wedge d\alpha + \dot{f} (\tau \wedge d\alpha + f e^{-t} \tau \wedge d\tau)$$

on  $\mathbb{R}P^3$  are positive for all  $t$ . For  $R$  sufficiently large this is the case because  $|\dot{f}|$  is bounded by  $2/R$  and  $\dot{f}$  has support in  $[t_0, t_0 + R]$ .  $\square$

**Remark 2.2.** The existence of a primitive of  $\eta$  follows by [10, Lemma 12.6] as well. In fact, in the case of a general connected base manifold  $Q$  the Gysin sequence implies that  $\eta$  admits a primitive if and only if the cohomology class of  $\eta$  is a multiple of the Euler class of  $Q$ . In other words,  $\eta$  admits a primitive if either the magnetic form is exact or  $Q$  is a surface different from a torus, cf. [10, Remark 12.7].

## 3. CAPACITY BOUNDS

Let  $U$  be an open relatively compact subset of the twisted cotangent bundle  $(T^*S^2, \omega_\sigma)$ . In view of Lemma 2.1 we can cut off the twisting outside a neighbourhood of  $\overline{U}$  so that the resulting symplectic form is standard on the complement of a co-disc bundle  $V$  of sufficiently large radius.

We provide  $\mathbb{CP}^1 \times \mathbb{CP}^1$  with the split symplectic form  $C(\omega_{\text{FS}} \oplus \omega_{\text{FS}})$ , where  $\omega_{\text{FS}}$  denotes the Fubini-Study form, so that the anti-diagonal is a Lagrangian 2-sphere. The co-disc bundle  $V$  equipped with the Liouville symplectic form embeds symplectically for  $C > 0$  sufficiently large. Pushing forward the cut off twisted symplectic form constructed in Lemma 2.1 we obtain a symplectic manifold  $(W, \omega)$  diffeomorphic to  $\mathbb{CP}^1 \times \mathbb{CP}^1$ , so that  $(U, \omega_\sigma)$  embeds symplectically. Therefore, in order to prove Theorem 1.2 it suffices to show that  $(W, \omega)$  has finite Hofer-Zehnder capacity.

Observe that the diagonal of  $\mathbb{CP}^1 \times \mathbb{CP}^1$  is a symplectically embedded 2-sphere of self-intersection 2 in  $(W, \omega)$ . Because the intersection form of  $\mathbb{CP}^1 \times \mathbb{CP}^1$  is even the symplectic manifold  $(W, \omega)$  is minimal. By the classification result [24, Corollary 1.6] of McDuff,  $(W, \omega)$  is symplectomorphic to  $\mathbb{CP}^1 \times \mathbb{CP}^1$  provided with a split symplectic form  $a\omega_{\text{FS}} \oplus b\omega_{\text{FS}}$ ,  $a, b > 0$ , cf. [18, Remark (1) on p. 6]. Using Lu's finiteness result [21, Theorem 1.21], which is based on work of Hofer-Viterbo [13], Floer-Hofer-Salamon [8], and Liu-Tian [19] (cf. McDuff-Slimowitz [26] and [25]), we obtain  $(a + b)\pi$  for the variant of the Hofer-Zehnder capacity of  $(W, \omega)$  that is defined by Frauenfelder-Ginzburg-Schlenk in [9]. According to [9, Appendix B] almost existence still is implied.

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